

Collective synchronization in the presence of reactive coupling and shear diversity

Ernest Montbrió¹ and Diego Pazó²

¹*Computational Neuroscience Group, Department of Information and Communication Technologies, Universitat Pompeu Fabra, 08003 Barcelona, Spain*

²*Instituto de Física de Cantabria (IFCA), CSIC-Universidad de Cantabria, 39005 Santander, Spain*

(Dated: September 23, 2011)

We analyze the synchronization dynamics of a model obtained from the phase reduction of the mean-field complex Ginzburg-Landau equation with heterogeneity. We present exact results that uncover the role of dissipative and reactive couplings on the synchronization transition when shears and natural frequencies are independently distributed. As it occurs in the purely dissipative case, an excess of shear diversity prevents the onset of synchronization, but this does not hold true if coupling is purely reactive. In this case the synchronization threshold turns out to depend on the mean of the shear distribution, but not on all the other distribution's moments.

PACS numbers: 05.45.Xt, 89.75.-k, 87.10.-e, 62.25.-g

Reaction-diffusion systems consisting of a large number of degrees of freedom display a rich variety of dynamical regimes that are important in a wide range of fields [1–3]. In particular, systems composed of many interacting aggregates of heterogeneous, self-oscillating elements, often show oscillations at the macroscopic level as a consequence of the collective synchronization of the individual oscillators [1, 4–6]. An appropriate model to study collective synchronization is the mean-field version of the complex Ginzburg-Landau equation (CGLE) with heterogeneity,

$$\dot{z}_j = z_j [1 + i(\omega_j + q_j) - (1 + iq_j)|z_j|^2] + \frac{K}{N}(1 + ic) \sum_{k=1}^N (z_k - z_j). \quad (1)$$

This equation describes an ensemble of $N \gg 1$ globally coupled limit-cycle oscillators, each defined by a complex variable $z_j \equiv \rho_j e^{i\theta_j}$. Every oscillator differs from the rest in the natural frequency of rotation ω_j and in the shear (or non-isochronicity) q_j , which measures how the frequency of rotation depends on the oscillator's amplitude ρ_j . Here we consider ω_j and q_j to be independent random variables, with a joint probability function $p(\omega, q) = g(\omega)h(q)$.

The oscillators are coupled via a diffusive coupling of strength K , which has both a real (dissipative) and an imaginary (reactive) components. In general a positive dissipative coupling drives the system to a more homogeneous state [7] (but see [8]). The effect of reactive coupling on synchronization is more intricate and strongly relies on the presence of shear $q_j \neq 0$ [9].

More than 30 years ago, the Kuramoto model (KM) was proposed as an analytically tractable system to study collective synchronization [10]. Since then it has become a paradigmatic model to explain temporal organization in a large variety of natural systems far from thermodynamic equilibrium [5, 11]. Under some approximations, the KM can be rigorously obtained from Eq. (1). Indeed, when the mutual coupling K of the oscillators is weak, a perturbation treatment permits to reduce Eq. (1) to a set of N equations for the phases

only [1],

$$\dot{\theta}_j = \omega_j + K(q_j - c) + KR[(1 + q_j c) \sin(\Psi - \theta_j) - (q_j - c) \cos(\Psi - \theta_j)], \quad (2)$$

where $R e^{i\Psi} = N^{-1} \sum_{k=1}^N e^{i\theta_k}$ is the complex order parameter. Originally, Kuramoto considered Eq. (1) without reactive coupling and without shear [10]. The resulting phase equation (2) with $c = q_j = 0$ is the well-known KM. Assuming constant shear, $q_j = \hat{q}$, model (2) is equivalent to the so-called Sakaguchi-Kuramoto model [12, 13]. This can be seen using the definition $\tan \beta_j = (q_j - c)/(1 + q_j c)$, with $|\beta_j| \leq \frac{\pi}{2}$, which permits to write Eq. (2) in the more compact form

$$\dot{\theta}_j = \omega_j + \frac{(1 + q_j c)K}{\cos \beta_j} [R \sin(\Psi - \theta_j - \beta_j) + \sin \beta_j].$$

As in the KM, the Sakaguchi-Kuramoto model shows a transition from incoherence to collective synchronization at large enough values of $K(1 + \hat{q}c)$. The synchronized solution can be obtained explicitly if $g(\omega)$ is a Lorentzian distribution.

We recently reported in [14] that, if shear is distributed according to a certain probability function $h(q)$, the onset of synchronization is prevented when the width of $h(q)$ exceeds a precise threshold. These results were obtained assuming purely dissipative coupling ($c = 0$), and are fully analytic if $g(\omega)$ and $h(q)$ are both Lorentzian. More recently [15] we allowed ω and q to be non-independent, but still considering $c = 0$.

Our first aim in this paper is to analyze the phase reduction (2) with reactive coupling ($c \neq 0$) and independent random variables ω_j and q_j . We will show that in this case, if $g(\omega)$ and $h(q)$ are both Lorentzian, the onset of synchronization is also prevented beyond a critical value of the width of $h(q)$. In the second part we address the case of purely reactive coupling, since it has physical relevance in the context of arrays of coupled nanomechanical oscillators [16–18], and in ion chains interacting via Coulomb forces [19]. We will demonstrate that, in this case, the synchronization's critical coupling becomes fully independent of the particular shape of the shear distribution $h(q)$. Finally we briefly discuss the

implications of this result for the KM with random coupling strengths recently studied by Hong and Strogatz [20, 21].

To analyze Eq. (2) we adopt the thermodynamic limit $N \rightarrow \infty$. This allows us to define a probability density function (PDF) for the phases $f(\theta, \omega, q, t)$, such that the complex order parameter $r = R e^{i\Psi}$ is

$$r(t) = \iint_{-\infty}^{\infty} \int_0^{2\pi} e^{i\theta} f(\theta, \omega, q, t) d\theta d\omega dq.$$

The evolution of Eq. (2) obeys the continuity equation

$$\partial_t f = -\partial_\theta \left(\{ \omega + K(q - c) + \frac{K}{2i} [r e^{-i\theta} C(1 - iq) - \text{c.c.}] \} f \right). \quad (3)$$

Here $C \equiv 1 + ic$, and c.c. stands for complex conjugate of the preceding term. Next we expand f in Fourier series as $f(\theta, \omega, q, t) = \frac{1}{2\pi} g(\omega) h(q) \sum_{l=-\infty}^{\infty} f_l(\omega, q, t) e^{il\theta}$, with $f_l = f_{-l}^*$, and $f_0 = 1$. Substituting the Fourier series into the continuity equation (3), we obtain the infinite set of integro-differential equations

$$\partial_t f_l = -il[\omega + K(q - c)]f_l + \frac{Kl}{2} [r^* C^* (1 + iq) f_{l-1} - r C (1 - iq) f_{l+1}]. \quad (4)$$

The next step is to assume that the asymptotic solutions of the model belong to the family of functions

$$f_l(\omega, q, t) = \alpha(\omega, q, t)^l, \quad (5)$$

a type of ansatz discovered by Ott and Antonsen [22–24]. This solution of Eq. (4) requires α to evolve according to

$$\partial_t \alpha = -i[\omega + K(q - c)]\alpha + \frac{K}{2} [r^* C^* (1 + iq) - r C (1 - iq) \alpha^2], \quad (6)$$

with

$$r^*(t) = \iint_{-\infty}^{\infty} g(\omega) h(q) \alpha(\omega, q, t) d\omega dq. \quad (7)$$

A considerable simplification is achieved if $g(\omega)$ and $h(q)$ are chosen to be Lorentzian PDFs

$$g(\omega) = \frac{\delta/\pi}{(\omega - \omega_0)^2 + \delta^2}; \quad h(q) = \frac{\gamma/\pi}{(q - q_0)^2 + \gamma^2}. \quad (8)$$

In this case the integral (7) can be solved closing the integrals at infinity and using the residue's theorem; notice that $g(\omega) = (2\pi i)^{-1}[(\omega - \omega_0 - i\delta)^{-1} - (\omega - \omega_0 + i\delta)^{-1}]$, likewise for $h(q)$. The important requirement is that the complex function α can be analytically continued from real ω and q into the complex planes $\omega = \omega_r + i\omega_i$ and $q = q_r + iq_i$, inside the integration contours.

It can be shown that α has no singularities in the lower half ω -plane [22]. Regarding the variable q , we follow the reasoning in [14] and find that α is analytic in the lower half complex q -plane for $K > 0$, and in the upper one for $K < 0$. However, now this holds true only if the order parameter satisfies [25]

$$R < R_\times = \frac{1}{\sqrt{1 + c^2}}. \quad (9)$$

We assume that states fulfilling this condition are correctly analyzed within this framework. As we show below, the numerical simulations fully confirm the validity of this assumption.

Therefore, using the residue's theorem, the integrals in (7) give

$$r^*(t) = \alpha(\omega = \omega^p, q = q^p, t) \equiv a(t), \quad (10)$$

where $\omega^p = \omega_0 - i\delta$ and $q^p = q_0 \mp i\gamma$ (for positive and negative K , respectively) correspond to the simple poles of the Lorentzian PDFs (8). The infinite set of ordinary differential equations (6) then simply reduces to the single ordinary differential equation with complex variable

$$\dot{a} = -i\omega^p a + \frac{K}{2} C(1 - iq^p)(1 - |a|^2)a.$$

As $a = R e^{-i\Psi}$, the equations for the order parameter inside the manifold defined by Eq. (5), read

$$\dot{R} = [-\delta + \frac{K}{2}(1 + cq_0 \mp \gamma)(1 - R^2)] R, \quad (11)$$

$$\dot{\Psi} = \omega_0 + \frac{K}{2} [q_0 - c(1 \mp \gamma)] (1 - R^2), \quad (12)$$

which we conjecture are the correct equations for the evolution of the order parameter, as far as condition (9) is fulfilled. From Eq. (11) we find that a synchronized solution bifurcates from incoherence at the critical coupling

$$K_c = \begin{cases} \frac{2\delta}{1 + q_0 c - \gamma} & \text{if } 1 + q_0 c > 0; \\ \frac{2\delta}{1 + q_0 c + \gamma} & \text{if } 1 + q_0 c < 0, \end{cases} \quad (13)$$

which only exists if

$$\gamma < \gamma_d = |1 + q_0 c|. \quad (14)$$

Otherwise, if $\gamma \geq \gamma_d$, incoherence becomes the only stable state for all K , see Fig. 1. This result extends the one found in [14] for $c = 0$ to ensembles of oscillators globally coupled via *both* dissipative and reactive coupling. However, as it is depicted in Fig. 1(a), now the region of stable synchronization is located at positive values of K only if

$$1 + q_0 c > 0. \quad (15)$$

This inequality is the well-known Benjamin-Feir-Newell criterion for the stability of plane waves in the homogeneous CGLE [1, 3, 13, 26, 27], which is valid in any dimension (infinite in the present case). Finally, we also find that the order parameter of the partially synchronized solution follows

$$R^2 = \frac{K - K_c}{K}, \text{ with } R < R_\times, \quad (16)$$

exactly as in the KM with Lorentzian $g(\omega)$ [1]. However, now this formula holds only up to R_\times , recall Eq. (9). The insets in Fig. 1 show numerical simulations that confirm the validity of Eq. (16). Remarkably, our numerical simulations indicate that R departs from Eq. (16) precisely when R exceeds R_\times .

Unfortunately, there is no straightforward theoretical extension of these results to more general distributions $g(\omega)$ and $h(q)$, but still some reasonable conjectures can be raised:

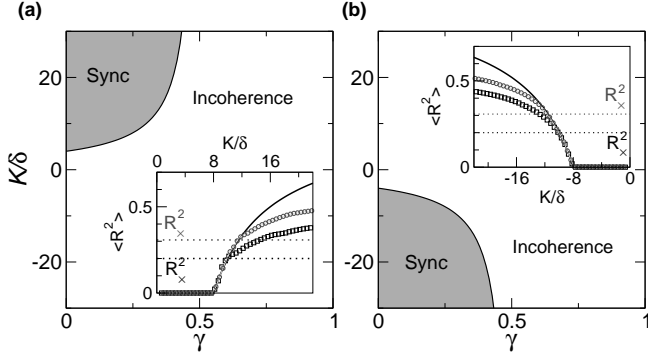


FIG. 1. Phase diagrams with boundary (13) for (a) $q_0 c = -\frac{1}{2}$, and (b) $q_0 c = -\frac{3}{2}$. Insets show the numerical results of the time averaged quantity R^2 vs. K/δ , with parameters $\gamma = \frac{1}{4}$ ($|K_c| = 8\delta$), $\delta = 0.1$, and $\omega_0 = 3$. $\{\omega_j, q_j\}_{j=1,\dots,N}$ were deterministically selected to represent the distribution (8) with $N = 14400$ oscillators. The data sets correspond to two different combinations of q_0 and c , with $c = 2$ ($R_\infty^2 = \frac{1}{3}$) and $c = \frac{3}{2}$ ($R_\infty^2 = \frac{4}{13}$). The solid line is Eq. (16).

- (i) The flip of the synchronization region when $1 + q_0 c$ changes sign—compare Figs. 1(a) and 1(b)—is a consequence of the Benjamin-Feir-Newell criterion (15) and it can be supposed to be a general feature of Eq. (2).
- (ii) Although it seems difficult to prove, the divergence of K_c at a critical value of the shear diversity [Eq. (14)] is likely to be a general property, as it happens in the purely dissipative case ($c = 0$) [14].
- (iii) For certain distributions and parameter values, and as a consequence of the persistence of a fully synchronized solution $\theta_j = \Psi$ located at $|K|g(\omega_0) \rightarrow \infty$, stable synchronization and incoherence should coexist at large K values (as it occurs for $c = 0$ and Gaussian distributions [14]). Note that infinitesimal perturbations obey $\delta\theta_j = \frac{K}{N}(1 + q_j c)[(1 - N)\delta\theta_j + \sum_{l \neq j} \delta\theta_l]$. The Jacobian matrix has always one trivial zero eigenvalue. If $c = 0$ and K positive, the remaining eigenvalues are negative, and the fixed point is stable irrespective of the width of $h(q)$. However, if $c \neq 0$ the fixed point becomes a *saddle* when the q_j 's exceed some degree of heterogeneity, and hence its continuation at finite K is not an attractor either. In sum, under a large enough heterogeneity of shear, incoherence should be the only attractor at all K values; however, if $c = 0$, synchronization coexists with incoherence at large enough K —provided $h(q)$ is not heavy-tailed, see [14].

For the remainder of this article, we will concentrate on the case of *purely reactive coupling*. Motivated by the dynamics of nanoscale mechanical oscillator arrays, this problem was analyzed in detail by Cross et al. [16, 17] with a coupling of the form $i\frac{\kappa}{N}\sum_k(z_k - z_j)$ in Eq. (1), and without shear diversity. To investigate the effect of shear diversity, we first write the phase model (2) without dissipative coupling. Substituting $c = \kappa/K$ in Eq. (2) and letting $K \rightarrow 0$ yields

$$\dot{\theta}_j = \omega_j - \kappa + \kappa R [q_j \sin(\Psi - \theta_j) + \cos(\Psi - \theta_j)], \quad (17)$$

where κ is now the total reactive coupling. Then, Eq. (13) suggests that in this limit the critical coupling is

$$\kappa_c = \frac{2\delta}{q_0}, \quad (18)$$

that remarkably depends on q_0 but is independent of the amount of heterogeneity γ .

The derivation of Eq. (18) is not rigorous because $R_\infty = 0$ in this limit, and condition (9) is not fulfilled. Therefore, to confirm the validity of Eq. (18) and to determine how this result generalizes to other distributions, next we perform the linear stability analysis of the incoherent state of Eq. (17) [28]. In the incoherent state all modes f_l , save the trivial one $f_0 = 1$, vanish. The equation for the Fourier modes, related to Eq. (4), is

$$\partial_t f_l = -il(\omega - \kappa)f_l + \frac{\kappa l}{2} [r^*(q - i)f_{l-1} - r(q + i)f_{l+1}].$$

Linearizing this equation about the incoherent state, we find that the only potentially unstable mode is the $l = 1$

$$\begin{aligned} \partial_t f_1 = & -i(\omega - \kappa)f_1 \\ & + \frac{\kappa}{2}(q - i) \iint_{-\infty}^{\infty} f_1(\omega', q', t) g(\omega') h(q') d\omega' dq'. \end{aligned}$$

Let $f_1(\omega, q, t) = b(\omega, q)e^{\lambda t}$, and neglect the trivial solution $b = 0$. Invoking self-consistency and separating λ into its real and imaginary parts ($\lambda = \lambda_r + i\lambda_i$) yields

$$\frac{2}{\kappa} = \iint_{-\infty}^{\infty} \frac{(q - i)[\lambda_r - i(\omega - \kappa + \lambda_i)]}{\lambda_r^2 + (\omega - \kappa + \lambda_i)^2} g(\omega) h(q) d\omega dq.$$

The interesting feature in the right hand side of this equation is that the integration over q is trivial and the result does *not* depend on the particular shape of $h(q)$. Performing the limit $\lambda_r \rightarrow 0^+$ to obtain the stability threshold κ_c yields

$$\frac{2}{\kappa_c} = (q_0 - i) \left[\pi g(\kappa_c - \lambda_i) - i \int_{-\infty}^{\infty} \frac{g(\omega)}{\omega - \kappa_c + \lambda_i} d\omega \right], \quad (19)$$

that only depends on $h(q)$ through its mean value q_0 (defined as principal value if required). Finally, splitting (19) into its real and imaginary parts we obtain a system of two equations for the unknowns κ_c and λ_i

$$(1 + q_0^2)\pi g(\kappa_c - \lambda_i) = \frac{2q_0}{\kappa_c}, \quad (20)$$

$$(1 + q_0^2) \int_{-\infty}^{\infty} \frac{g(\omega)}{\omega - \kappa_c + \lambda_i} d\omega = -\frac{2}{\kappa_c}. \quad (21)$$

These equations can be solved for Lorentzian $g(\omega)$, and indeed we recover the boundary (18). However, note that now this result is stronger, since we have not imposed any constraint on the shape of $h(q)$. Here $h(q)$ can be *any* distribution of mean q_0 . Additionally, an explicit value for κ_c can be easily obtained from Eq. (20) if $g(\omega)$ is a uniform (top-hat) distribution. These results for Lorentzian and uniform $g(\omega)$ are in agreement with those obtained in [17] with $h(q) = \delta(q - q_0)$.

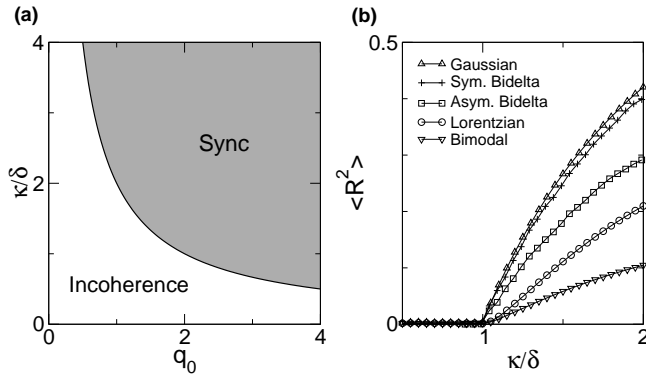


FIG. 2. (a) Phase diagram with boundary (18) in the case of purely reactive coupling and $g(\omega)$ Lorentzian. (b) Time averages $\langle R^2 \rangle$ obtained from numerical simulations with $q_0 = 2$, $\delta = 1$, $N = 40000$, and for different distributions of q : Gaussian with variance $\nu^2 = 1$, symmetric bidelta: $h(q) = \frac{1}{2}\delta(q-3) + \frac{1}{2}\delta(q-1)$, asymmetric bidelta: $h(q) = \frac{3}{4}\delta(q-3) + \frac{1}{4}\delta(q+1)$, Lorentzian with $\gamma = 1$, and bimodal: $h(q) = \frac{1/(2\pi)}{[q-(q_0-5/2)]^2+1} + \frac{1/(2\pi)}{[q-(q_0+5/2)]^2+1}$. In all cases the critical coupling is at $\kappa_c/\delta = 1$, as predicted by Eq. (18).

This confirms that the phase equation is indeed a good approximation of the amplitude equation in the limit of weak coupling and narrow frequency distributions.

Figure 2(a) displays a phase diagram with the boundary (18), and Fig. 2(b) shows the time average $\langle R^2 \rangle$ obtained from numerical simulations using different distributions $h(q)$ with common q_0 values. As expected, in Fig. 2(b) the tran-

sition between incoherence and synchronization occurs at the same value of κ/δ irrespective of the distribution type.

Finally we point out an interesting similarity between Eq. (17) and the model recently studied by Hong and Strogatz [20, 21], which in our notation reads: $\dot{\theta}_j = \omega_j + q_j R \sin(\Psi - \theta_j)$. Note that here q_j acts as a distributed coupling strength. Performing a stability analysis like we did above, we obtain that the stability border of incoherence satisfies $2 = \pi q_0 g(\omega_0)$, if $g(\omega)$ is unimodal and symmetric. Again, we obtain a formula that depends on the mean of $h(q)$, but not on its shape. This result reproduces the classical Kuramoto relation for uniform all-to-all coupling [$h(q) = \delta(q - q_0)$], and the critical point found in Eq. (12) of [20] for Lorentzian $g(\omega)$ and bidelta $h(q)$.

In conclusion, we have reported on exact results that extend the phase models of Kuramoto and Sakaguchi [10, 12] to situations where shear is distributed. In contrast to the recent work [14], here the coupling is not purely dissipative but also contains a reactive component c . In this case we also find that shear diversity prevents the onset of collective synchronization, but the Benjamin-Feir-Newell criterion determines now if the region of synchronization is located at positive or negative values of K . Finally, we have obtained a remarkable result when the coupling is purely reactive: the stability threshold of incoherence depends on the mean shear q_0 , while the shear diversity becomes irrelevant.

Financial support from the Ministerio de Ciencia e Innovación (Spain) under Projects No. FIS2009-12964-C05-05 and No. SAF2010-16085 is acknowledged.

-
- [1] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer-Verlag, Berlin, 1984).
 - [2] J. D. Murray, *Mathematical biology, Interdisciplinary applied mathematics* (Springer, New York, 2003).
 - [3] M. C. Cross and P. C. Hohenberg, *Rev. Mod. Phys.* **65**, 851 (1993).
 - [4] A. T. Winfree, *The Geometry of Biological Time* (Springer, New York, 1980).
 - [5] A. S. Pikovsky, M. G. Rosenblum, and J. Kurths, *Synchronization, a Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001).
 - [6] S. H. Strogatz, *Sync: The emerging science of spontaneous order*. (Hyperion Press, New York, 2003).
 - [7] P. C. Matthews, R. E. Mirollo, and S. H. Strogatz, *Physica D* **52**, 293 (1991).
 - [8] H. Daido and K. Nakanishi, *Phys. Rev. Lett.* **96**, 054101 (2006).
 - [9] See [1, 5, 29] for the case of two coupled Stuart-Landau oscillators, and [13, 16, 17, 26] for the mean-field CGLE.
 - [10] Y. Kuramoto, in *International Symposium on Mathematical Problems in Theoretical Physics*, Vol. 39 of *Lecture Notes in Physics*, edited by H. Araki (Springer, Berlin, 1975), pp. 420–422.
 - [11] J. A. Acebrón *et al.*, *Rev. Mod. Phys.* **77**, 137 (2005).
 - [12] H. Sakaguchi and Y. Kuramoto, *Prog. Theor. Phys.* **76**, 576 (1986).
 - [13] N. Nakagawa and Y. Kuramoto, *Prog. Theor. Phys.* **89**, 313 (1993).
 - [14] E. Montbrió and D. Pazó, *Phys. Rev. Lett.* **106**, 254101 (2011).
 - [15] D. Pazó and E. Montbrió, *EPL* **95**, 60007 (2011).
 - [16] M. C. Cross, A. Zumdieck, R. Lifshitz, and J. L. Rogers, *Phys. Rev. Lett.* **93**, 224101 (2004).
 - [17] M. C. Cross, J. L. Rogers, R. Lifshitz, and A. Zumdieck, *Phys. Rev. E* **73**, 036205 (2006).
 - [18] S.-B. Shim, M. Imboden, and P. Mohanty, *Science* **316**, 95 (2007).
 - [19] T. E. Lee and M. C. Cross, *Phys. Rev. Lett.* **106**, 143001 (2011).
 - [20] H. Hong and S. H. Strogatz, *Phys. Rev. Lett.* **106**, 054102 (2011).
 - [21] H. Hong and S. H. Strogatz, arXiv:1108.6117 (2011).
 - [22] E. Ott and T. M. Antonsen, *Chaos* **18**, 037113 (2008).
 - [23] E. Ott and T. M. Antonsen, *Chaos* **19**, 023117 (2009).
 - [24] E. Ott, B. R. Hunt, and T. M. Antonsen, *Chaos* **21**, 025112 (2011).
 - [25] In the semicircular path at infinity ($|q| \rightarrow \infty$) parametrized by ϑ , $q = |q|e^{i\vartheta}$, the evolution equation of $|\alpha|$ evaluated at $|\alpha| = 1$ yields at leading order $\partial_t |\alpha| = K(1 - R|C| \cos \chi)|q| \sin \vartheta$, where $\alpha = |\alpha|e^{-i\psi}$, $C = |C|e^{i\zeta}$, and $\chi = \psi(q, t) - \Psi(t) - \zeta$. Thus, the condition $\partial_t |\alpha| < 0$ assuring that Eq. (5) remains finite, is fulfilled if $R < R_\times = |C|^{-1}$, in $\vartheta \in (0, \pi)$ for $K < 0$, and in $\vartheta \in (-\pi, 0)$ for $K > 0$.
 - [26] V. Hakim and W. J. Rappel, *Phys. Rev. A* **46**, R7347 (1992).
 - [27] I. S. Aranson and L. Kramer, *Rev. Mod. Phys.* **74**, 99 (2002).
 - [28] S. H. Strogatz and R. E. Mirollo, *J. Stat. Phys.* **63**, 613 (1991).
 - [29] D. G. Aronson, G. B. Ermentrout, and N. Kopell, *Physica D* **41**,

403 (1990).